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Best Polynomial Approximation to Certain Entire Functions

A. R. REDDY

Department of Mathematics, The University of Toledo, Toledo, Ohio 43606

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Let $f(x)$ be a real-valued continuous function on $[-1, 1]$, and let

$$E_n(f) \equiv \inf_{p \in \pi_n} \| f - p \|, \qquad n = 0, 1, 2,...,$$

where the norm is the uniform norm on $[-1, 1]$ and π_n denotes the set of all polynomials with real coefficients of degree at most n. Bernstein [1, p. 118] has shown that

$$\lim_{n \to \infty} E_n^{1/n}(f) = 0 \tag{1}$$

if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function.

Let $f(z)$ be an entire function, and let

$$M(r) = \max_{|z|=r} |f(z)|;$$

then the order ρ, lower order λ, type τ and lower type ω of $f(z)$ are defined by

$$\begin{aligned} \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(r)}{\log r} &= \begin{matrix} \rho \\ \lambda \end{matrix} \qquad (0 \leqslant \lambda \leqslant \rho \leqslant \infty), \\ \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(r)}{r^\rho} &= \begin{matrix} \tau \\ \omega \end{matrix} \qquad \begin{pmatrix} 0 < \rho < \infty \\ 0 \leqslant \omega \leqslant \tau \leqslant \infty \end{pmatrix}. \end{aligned} \tag{2}$$

S. N. Bernstein [1, p. 114] proved that there exist (finite) constants $\rho > 0$, $0 \leqslant \tau < \infty$ such that

$$\lim_{n \to \infty} \sup n^{1/\rho} E_n^{1/n}(f) \tag{3}$$

is finite if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ρ and type τ.

Recently, Varga [7, Th. 1] has proved that

$$\rho = \lim_{n\to\infty} \sup \frac{n \log n}{\log[E_n(f)]^{-1}} \tag{4}$$

satisfies $0 \leqslant \rho < \infty$ if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ρ.

The results of Bernstein and Varga give us the clue that the rate at which $E_n^{1/n}(f)$ tends to zero depends on the order and type of an entire function.

To deal with functions of infinite order, we introduce the following classification[1]. We shall assume for such a function, that there exists a positive integer $k \geqslant 2$, for which

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_{k+1}M(r)}{\log r} = \begin{matrix}\rho(k)\\ \lambda(k)\end{matrix} \tag{5}$$

are finite and positive. Here we have used the familiar notation

$$l_k x = \underbrace{\log \log \cdots \log}_{k \text{ times}} x \qquad (k = 1, 2, 3,...).$$

Note that $l_k x > 0$ for all sufficiently large x. An entire function $f(z)$ with $\rho(k-1) = \infty$ and $\rho(k) < \infty$ is called an entire function of index k. Note that $\rho(k)$ and $\lambda(k)$ generalize ρ and λ of (2), which correspond to $k = 1$. If $\rho(k)$ is positive and finite, we can associate with it the functionals $\tau(k,f) = \tau(k)$ and $\omega(k,f) = \omega(k)$, defined by

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_k M(r)}{r^{\rho(k)}} = \begin{matrix}\tau(k)\\ \omega(k)\end{matrix}. \tag{6}$$

Another classification has been introduced for the class of (transcendental) entire functions of order 0, by means of the logarithmic order ρ_l and the corresponding lower order λ_l. They are defined thus:

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log M(r)}{\log \log r} = \begin{matrix}\rho_l\\ \lambda_l\end{matrix} \tag{7}$$

$$(1 \leqslant \lambda_l \leqslant \rho_l \leqslant \infty).$$

If ρ_l is greater than one and finite, we can define the logarithmic type τ_l of f and the corresponding lower type ω_l, by

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{(\log r)^{\rho_l}} = \begin{matrix}\tau_l\\ \omega_l\end{matrix}. \tag{8}$$

[1] A slightly different classification has been studied in [8].

The following result is also due to S. N. Bernstein [2, p. 77, Theorem 59]: Let $f(x)$ and $g(x)$ be real-valued functions with continuous $(n+1)$-th derivative on the interval $[-1, 1]$ and let

$$|f^{n+1}(x)| \leqslant g^{n+1}(x) \text{ throughout } [-1, 1].$$

Then

$$E_n(f) \leqslant E_n(g). \tag{9}$$

Bernstein [1, p. 116] also proved that, if $f(\mathscr{Z}) = \sum_{k=0}^{\infty} a_k \mathscr{Z}^k$ is an entire function satisfying

$$\lim_{n\to\infty} n^{1/2} \mid a_n \mid^{1/n} = 0, \tag{10}$$

then there exists a subsequence $(n_1, n_2, ...)$ of $(0, 1, 2, ...)$ such that $| a_{n_\mu^{+1}} | \neq 0$ and

$$\lim_{\mu\to\infty} \frac{2^n \mu E_{n_\mu}(f)}{| a_{n_\mu^{+1}} |} = 1. \tag{11}$$

One of the purposes of this paper is to investigate under what conditions on $E_n(f)$, f is the restriction to $[-1, 1]$ of an entire function of order < 1 or order 1 and type 0. We study also how $E_n(f)/E_{n+1}(f)$ is related to ρ, ρ_l, $\rho(k)$, τ, τ_l, and $\tau(k)$. Further, we prove the last result of Bernstein for a wider class of functions, namely, for entire functions of perfectly regular growth [6, p. 44]. For entire functions of order 0 or ∞, we study the growth of $E_n(f)$ and $| a_n |$. Furthermore, we study the following problem, related to the well-known results of Bernstein and Shohat [5, p. 379]. Given two entire functions $f(z) = \sum_{K=0}^{\infty} a_K z^K$, $g(z) = \sum_{k=0}^{\infty} b_k z^k$, with respective orders and types ρ_f, λ_f, τ_f, ω_f, and ρ_g, λ_g, τ_g, ω_g, what is the relation between $E_n(f)/E_n(g)$ and these orders and types? The bounds we obtain here are much sharper than those of Bernstein and Shohat.

Entire functions of regular growth

DEFINITION. An entire function f is of regular growth [6, p. 41] if

$$\lim_{r\to\infty} \frac{\log \log M(r)}{\log r} = \rho$$

exists. A necessary and sufficient condition that an entire function f be of regular growth is that the coefficients a_n satisfy, for every $\epsilon > 0$, the inequality

$$| a_n |^{1/n} < n^{-1/(\rho+\epsilon)}, \text{ for all large } n,$$

and that there exist a strictly increasing sequence $\{n_p\}_1^\infty$ of positive integers such that

$$\lim_{p\to\infty} \frac{\log n_{p+1}}{\log n_p} = 1 \qquad \text{and}$$
$$\lim_{p\to\infty} \frac{n_p \log n_p}{\log | 1/a_{n_p} |} = \rho. \tag{12}$$

Entire functions of perfectly regular growth

It is known that an entire function is of perfectly regular growth (ρ, τ), $0 < \rho < \infty, 0 < \tau < \infty$, if and only if, given $\epsilon > 0$, there exists an $n_0(\epsilon)$ such that

$$\frac{n}{\rho e} | a_n |^{\rho/n} < \tau + \epsilon, \qquad \text{for} \quad n \geqslant n_0(\epsilon)$$

and there exists a strictly increasing sequence $\{n_p\}_1^\infty$ of positive integers such that

$$\lim_{p\to\infty} \frac{n_{p+1}}{n_p} = 1 \qquad \text{and}$$
$$\lim_{p\to\infty} \frac{n_p}{\rho e} | a_{n_p} |^{\rho/n_p} = \tau. \tag{13}$$

We shall need several lemmas.

LEMMA 1. *Let* $f(z) = \sum_{k=0}^\infty a_k z^k$ *be an entire function of index* k, *order* $\rho(k)$ *and lower order* $\lambda(k)$ $(0 \leqslant \lambda(k) \leqslant \rho(k) < \infty)$. *Then*

$$\liminf_{n\to\infty} \frac{1_k n}{\log | a_n/a_{n+1} |} \leqslant \liminf_{n\to\infty} \frac{n \cdot 1_k n}{\log | 1/a_n |} \leqslant \lambda(k)$$
$$\leqslant \rho(k) = \limsup_{n\to\infty} \frac{n \cdot 1_k n}{\log | 1/a_n |} \leqslant \lim_{n\to\infty} \frac{1_k n}{\log | a_n/a_{n+1} |}.$$

Proof. The result is known for the case $k = 1$ [4, p. 1046]. The middle inequality is known for $k \geqslant 1$ [3, Lemmas 1 and 2A]. The extreme inequalities, when $k \geqslant 2$, follow as for the case $k = 1$, and hence we omit the proof.

LEMMA 2. *Let* $f(z) = \sum_{k=0}^\infty a_k z^k$ *be an entire function of index* k, *with lower order* $\lambda(k)$, *such that* $| a_n |/| a_{n+1} |$ *is nondecreasing for* $n \geqslant n_0$. *Then*

$$\lambda(k) = \liminf_{n\to\infty} \frac{n \cdot 1_k n}{\log | 1/a_n |} = \liminf_{n\to\infty} \frac{1_k n}{\log | a_n/a_{n+1} |},$$
$$\rho(k) = \limsup_{n\to\infty} \frac{\log n}{\log | a_n/a_{n+1} |}.$$

Proof. This lemma is known for when $k = 1$ [4, p. 1047]. The proof, when $k \geqslant 2$, proceeds as in the case $k = 1$, using [3, Lemmas 1, 2B].

LEMMA 3. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function of logarithmic order ρ_l and logarithmic lower order λ_l. Then*

$$\liminf_{n\to\infty} \frac{\log n}{\log\log | a_n/a_{n+1} |} \leqslant \liminf_{n\to\infty} \frac{\log n}{\log\{1/n \log | 1/a_n |\}}$$
$$\leqslant \lambda_l - 1 \leqslant \rho_l - 1 = \limsup_{n\to\infty} \frac{\log n}{\log\{1/n \log | 1/a_n |\}}$$
$$\leqslant \limsup_{n\to\infty} \frac{\log n}{\log\log | a_n/a_{n+1} |}.$$

Proof. See [3, Lemmas 5, 6A].

LEMMA 4. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be entire function of order ρ_l and lower order $\lambda_l (1 \leqslant \lambda_l \leqslant \rho_l < \infty)$ such that $| a_n/a_{n+1} |$ is nondecreasing for $n \geqslant n_0$. Then*

$$\lambda_l - 1 \geqslant \liminf_{n\to\infty} \frac{\log n}{\log\{1/n \log | 1/a_n |\}} = \liminf_{n\to\infty} \frac{\log n}{\log\log | a_n/a_{n+1} |},$$
$$\rho_l - 1 = \limsup_{n\to\infty} \frac{\log n}{\log\log | a_n/a_{n+1} |}.$$

Proof. See [3, Lemmas 5, 6B].

THEOREMS

THEOREM 1. *Let $f(x)$ be a real-valued continuous function defined on $[-1, 1]$. Then*

$$\lim_{n\to\infty} n E_n^{1/n}(f) = 0 \tag{14}$$

if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function of either order < 1, or of order 1 and type 0.

Proof. If $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ρ and type τ, then it is known [3, Theorems 1, 3; 7, Theorem] that

$$\limsup_{n\to\infty} \frac{n \log n}{\log[1/E_n(f)]} = \rho, \qquad \limsup_{n\to\infty} \frac{n}{\rho e} E_n^{\rho/n}(f) = \frac{\tau}{2^\rho}$$

and hence

$$E_n^{1/n}(f) \leqslant n^{-1/(\rho+\epsilon)}, \qquad \text{for} \quad n \geqslant n(\epsilon).$$

Therefore,

$$nE_n^{1/n}(f) \leqslant n^{1-\frac{1}{\rho+\epsilon}}, \qquad \text{for} \quad n \geqslant n(\epsilon).$$

Hence, if $\rho < 1$

$$\lim_{n\to\infty} nE_n^{1/n}(f) = 0.$$

If $\rho = 1$ and $\tau = 0$, we have $\lim_{n\to\infty} nE_n^{1/n} = 0$, by the second equality of this proof. On the other hand, if $\lim_{n\to\infty} nE_n^{1/n}(f) = 0$, then $\lim_{n\to\infty} E_n^{1/n} = 0$, which indicates, because of Bernstein's theorem, that $f(x)$ is the restriction to $[-1, 1]$ of an entire function. We have to show that the order of this entire function is either less than one or one, with type 0. One can verify that

$$\limsup_{n\to\infty} \frac{n \log n}{\log\{1/E_n(f)\}} \leqslant 1.$$

Hence, either $\rho < 1$, or $\rho = 1$ and the type is zero.

THEOREM 2. *Let $f(x)$ be a real-valued continuous function defined on $[-1, 1]$. If $f(x)$ is the restriction to $[-1, 1]$ of an entire function of index k, order $\rho(k)$ and lower order $\lambda(k)$, then*

$$\liminf_{n\to\infty} \frac{l_k n}{\log[E_n(f)/E_{n+1}(f)]} \leqslant \liminf_{n\to\infty} \frac{n \cdot l_k n}{\log[1/E_n(f)]} \leqslant \lambda(k)$$
$$\leqslant \rho(k) \leqslant \limsup_{n\to\infty} \frac{l_k n}{\log[E_n(f)/E_{n+1}(f)]}.$$

Proof. Assume that $f(x)$ has an extension $f(z)$ which is an entire function of index k with lower order $\lambda(k)$. Then it is known [3, (11), (17)] that

$$M\left(\frac{\sigma^2 - 1}{2\sigma}\right) \leqslant B(\sigma) \leqslant M\left(\frac{\sigma^2 + 1}{2\sigma}\right), \tag{15}$$

$$B(\sigma) \leqslant C'\sigma H(\sigma) \leqslant C''\sigma(\sigma + \eta)\, B(\sigma + \eta), \tag{16}$$

where C', C'', η are constants and $B(\sigma)$ is the maximum of $|f(z)|$ on E_σ $(\sigma > 1)$, the closed interior of the ellipse with foci ± 1, major semiaxis $(\sigma^2 + 1/2\sigma)$ and minor semiaxis $(\sigma^2 - 1/2\sigma)$. It is also known [3, (18)] that

$$\begin{matrix}\rho(k,j)\\ \lambda(k,j)\end{matrix} \equiv \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_{k+j}M(\sigma)}{l_{j+1}\sigma} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_{k+j}B(\sigma)}{l_{j+1}\sigma}$$
$$= \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_{k+j}H(\sigma)}{l_{j+1}\sigma} \qquad \left(H(\sigma) = \sum_{n=0}^{\infty} E_n \sigma^n\right) \tag{17}$$

for any $k \geqslant 1, j \geqslant 1$. Now, applying Lemma 1 to $H(\sigma)$ of [3, (16)], we have the required result.

THEOREM 3. *Let $f(x)$ be a real-valued continuous function on $[-1, 1]$. Then if $f(x)$ is the restriction to $[-1, 1]$ of an entire function of index k, order $\rho(k)$ and lower order $\lambda(k)$ such that $E_n(f)/E_{n+1}(f)$ is nondecreasing for $n \geqslant n_0$,*[2] *then*

$$\lambda(k) = \liminf_{n\to\infty} \frac{n \cdot l_k n}{\log[1/E_n(f)]} = \liminf_{n\to\infty} \frac{l_k n}{\log[E_n(f)/E_{n+1}(f)]},$$
$$\rho(k) = \limsup_{n\to\infty} \frac{l_k n}{\log[E_n(f)/E_{n+1}(f)]}. \tag{18}$$

Proof. We apply Lemma 2 to $H(\sigma)$.

THEOREM 4. *Let $f(x)$ be a real-valued continuous function defined on $[-1, 1]$ which is the restriction to $[-1, 1]$ of an entire function $f(z)$ of logarithmic order ρ_l and logarithmic lower order λ_l. Then*

$$\liminf_{n\to\infty} \frac{\log n}{\log\log[E_n(f)/E_{n+1}(f)]} \leqslant \liminf_{n\to\infty} \frac{\log n}{\log\{1/n \log[1/E_n(f)]\}}$$
$$\leqslant \lambda_l - 1 \leqslant \rho_l - 1$$
$$\leqslant \limsup_{n\to\infty} \frac{\log n}{\log\log[E_n(f)/E_{n+1}(f)]}.$$

Proof. We have from (17)

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_2 H(\sigma)}{l_2 \sigma} = \begin{matrix}\rho_l\\ \lambda_l\end{matrix}. \tag{19}$$

One deduces the required result from (19) by applying Lemma 3 to $H(\sigma)$.

THEOREM 5. *Let $f(x)$ be real-valued continuous function on $[-1, 1]$ which is the restriction to $[-1, 1]$ of an entire function $f(z)$ of logarithmic order ρ_l and logarithmic lower order λ_l, such that $E_n(f)/E_{n+1}(f)$ is nondecreasing for $n \geqslant n_0$. Then*

$$\lambda_l - 1 = \liminf_{n\to\infty} \frac{\log n}{\log\{1/n \log[1/E_n(f)]\}}$$
$$= \liminf_{n\to\infty} \frac{\log n}{\log\log[E_n(f)/E_{n+1}(f)]},$$
$$\rho_l - 1 = \limsup_{n\to\infty} \frac{\log n}{\log\log[E_n(f)/E_{n+1}(f)]}.$$

[2] The functions $f(x) = e^x$, $g(x) = \cos x$ satisfy this property (cf. the example following Theorem 12).

Proof. It is known from (17) that

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{l_2 H(\sigma)}{l_2 \sigma} = \frac{\rho_l}{\lambda_l}.$$

Applying Lemma 4 to $H(\sigma)$, we have the result.

THEOREM 6. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function of positive order and regular growth with real a_k's. Then there are integers $1 \leqslant n_1 < n_2 < \cdots$ such that*

$$\lim_{p\to\infty} \frac{\log E_{n_p}}{\log |a_{n_p}|} = 1. \tag{20}$$

Proof. Since $f(z)$ is of regular growth, we have from [3], Theorem 1, and the existence of integers $1 \leqslant n_1 < n_2 < \cdots$ satisfying

$$\lim_{p\to\infty} \frac{\log n_{p+1}}{\log n_p}\Big) = 1,$$

the equalities

$$\lim_{p\to\infty} \frac{n_p \log n_p}{\log |1/a_{n_p}|} = \rho = \lim_{p\to\infty} \frac{n_p \log n_p}{\log(1/E_{n_p})}. \tag{21}$$

By (21),

$$\begin{aligned} \frac{n_p \log n_p}{\rho + \epsilon} &\leqslant \log \frac{1}{E_{n_p}} \leqslant \frac{n_p \log n_p}{\rho - \epsilon} \qquad \text{for} \quad p \geqslant p_0(\epsilon) \\ \frac{n_p \log n_p}{\rho + \epsilon} &\leqslant \log \frac{1}{|a_{n_p}|} \leqslant \frac{n_p \log n_p}{\rho - \epsilon} \qquad \text{for} \quad p \geqslant p_1(\epsilon). \end{aligned} \tag{22}$$

From (22), for a suitable ϵ',

$$1 - \epsilon' \leqslant \frac{\log E_{n_p}}{\log |a_{n_p}|} \leqslant 1 + \epsilon', \qquad \text{for} \quad p \geqslant \max(p_0, p_1).$$

Hence,

$$\lim_{p\to\infty} \frac{\log E_{n_p}}{\log |a_{n_p}|} = 1.$$

THEOREM 7. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function of perfectly regular growth with real a_k's. Then, for some sequence of integers $\{n_p\}_{p=1}^{\infty}$, $1 \leqslant n_1 < n_2 \cdots$, we have*

$$\lim_{p\to\infty} \left(\frac{E_{n_p}}{|a_{n_p}|}\right)^{1/n_p} = \frac{1}{2}. \tag{23}$$

Proof. It is known [3, Theorem 3] that if $f(z)$ is of perfectly regular growth, then

$$\lim_{n\to\infty} \frac{n}{\rho e} |a_n|^{\rho/n} = \lim_{n\to\infty} 2^\rho \frac{n}{\rho e} E_n^{\rho/n} = \tau.$$

It is also known [6, p. 44] that there exists a sequence $\{n_p\}_1^\infty$ of positive integers, $n_p \to \infty$, such that

$$\lim_{p\to\infty} \frac{n_{p+1}}{n_p} = 1$$

and

$$\lim_{p\to\infty} \frac{n_p}{\rho e} |a_{n_p}|^{\rho/n_p} = \tau = \lim_{p\to\infty} 2^\rho \frac{n_p}{\rho e} \{E_{n_p}(f)\}^{\rho/n_p}. \tag{24}$$

From (24) one has

$$\rho e(\tau - \epsilon) 2^{-\rho} \leqslant n_p E_{n_p}^{\rho/n_p} \leqslant (\tau + \epsilon) \rho e 2^{-\rho} \qquad \text{for} \quad p \geqslant p_0(\epsilon)$$

and

$$\rho e(\tau - \epsilon_1) \leqslant n_p |a_{n_p}|^{\rho/n_p} \leqslant (\tau + \epsilon_1) \rho e \qquad \text{for} \quad p \geqslant p_1(\epsilon_1),$$

and hence

$$\lim_{p\to\infty} (E_{n_p}/|a_{n_p}|)^{1/n_p} = \tfrac{1}{2}.$$

THEOREM 8. *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ (a_k real) is an entire function of index k, order $\rho(k)$ and corresponding lower order $\lambda(k)$ such that $|a_n/a_{n+1}|$ and $E_n(f)/E_{n+1}(f)$ are nondecreasing for $n \geqslant n_0$ and $n \geqslant n_1$, respectively, then*

$$\frac{\lambda(k)}{\rho(k)} \leqslant \liminf_{n\to\infty} \frac{\log E_n(f)}{\log |a_n|} \leqslant 1 \leqslant \limsup_{n\to\infty} \frac{\log E_n(f)}{\log |a_n|} \leqslant \frac{\rho(k)}{\lambda(k)}.$$

Remark. There exists an entire function $f(z) = \sum_{k=0} a_k z^k$ for which $|a_n/a_{n+1}|$ is nondecreasing and $\rho > \lambda$ ([4], p. 1047).

Proof of Theorem 8. It is known from [3, Lemmas 1, 2, Theorems 1, 2] and the fact that $E_n(f)/E_{n+1}(f)$ and $|a_n/a_{n+1}|$ are nondecreasing for $n \geqslant n_0'$ that

$$\lim_{n\to\infty} \frac{\sup}{\inf} \frac{nl_k n}{\log |1/a_n|} = \frac{\rho(k)}{\lambda(k)},$$

$$\lim_{n\to\infty} \frac{\sup}{\inf} \frac{nl_k n}{\log[1/E_n(f)]} = \frac{\rho(k)}{\lambda(k)}. \tag{25}$$

By using (25) and

$$\frac{\log E_n(f)}{\log a_n} = \frac{\log(1/E_n(f)]}{\log(1/a_n)} \cdot \frac{nl_k n}{nl_k n}, \tag{26}$$

we have

$$\begin{aligned}
&\varliminf_{n\to\infty} \frac{nl_k n}{\log |1/a_n|} \varliminf_{n\to\infty} \frac{\log[1/E_n(f)]}{nl_k n} \\
&\leqslant \varliminf_{n\to\infty} \frac{\log E_n(f)}{\log |a_n|} \leqslant \varlimsup_{n\to\infty} \frac{\log[1/E_n(f)]}{nl_k n} \varliminf_{n\to\infty} \frac{nl_k n}{\log 1/a_n} \\
&\leqslant \varlimsup_{n\to\infty} \frac{\log E_n(f)}{\log |a_n|} \leqslant \varlimsup_{n\to\infty} \frac{\log[1/E_n(f)]}{nl_k n} \varlimsup_{n\to\infty} \frac{nl_k n}{\log |1/a_n|},
\end{aligned}$$

and the result follows.

THEOREM 9. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ (a_k real) be an entire function of index k, order $\rho(k)$ and types $\tau(k)$, $\omega(k)$ $(0 \leqslant \omega(k) < \infty)$. Assume that $E_n(f)/E_{n+1}(f)$ and $|a_n|/|a_{n+1}|$ are non-decreasing, for $n \geqslant n_1$. Then for $k \geqslant 1$,*

$$\left(\frac{\omega(k)}{\tau(k)}\right)^{1/\rho(k)} \leqslant \varliminf_{n\to\infty} \frac{2E_n^{1/n}(f)}{|a_n|^{1/n}} \leqslant 1 \leqslant \varlimsup_{n\to\infty} \frac{2E_n^{1/n}(f)}{|a_n|^{1/n}} \leqslant \left(\frac{\tau(k)}{\omega(k)}\right)^{1/\rho(k)}. \tag{27}$$

Remark. The entire function $f(z) = \sum_{n=1}^{\infty} (\log n/n)^{\rho/n} z^n$ $(\rho > 0)$ has order ρ and type ∞. For this function f we can use Theorem 8 to relate $E_n(f)$ and $|a_n|$.

Proof of Theorem 9. From [3, Lemma 3 and Theorem 3] and the monotonicity of $E_n(f)/E_{n+1}(f)$ and $|a_n|/|a_{n+1}|$ we have, if $k = 1$,

$$\limsup_{n\to\infty} \frac{n}{\rho e} |a_n|^{\rho/n} = \limsup_{n\to\infty} \frac{n}{\rho e} E_n^{\rho/n} 2^{\rho} = \tau. \tag{28}$$

For any $k \geqslant 2$ (cf. [3]),

$$\limsup_{n\to\infty} l_{k-1}n \cdot |a_n|^{\rho(k)/n} = \limsup_{n\to\infty} l_{k-1}n \cdot E_n^{\rho(k)/n} 2^{\rho(k)} = \tau(k). \tag{29}$$

By (28), we have

$$\varliminf_{n\to\infty}\left(\frac{2E_n^{1/n}(f)}{|a_n|^{1/n}}\right) = \varliminf_{n\to\infty}\left(\left(\frac{n}{\rho e}\right)^{1/\rho}\frac{2E_n^{1/n}(f)}{|a_n|^{1/n}}\left(\frac{\rho e}{n}\right)^{1/\rho}\right)$$
$$\leqslant \varliminf_{n\to\infty} 2\left(\frac{n}{\rho e}\right)^{1/\rho} E_n^{1/n}(f)\ \varlimsup_{n\to\infty}\frac{1}{(n/((\rho e)^{1/\rho}|a_n|^{1/n}}$$
$$\leqslant \left(\frac{\omega(1)}{\omega(1)}\right)^{1/\rho} = 1.$$

Similarly, we can show that

$$\varliminf_{n\to\infty}\frac{2E_n^{1/n}(f)}{|a_n|^{1/n}} \geqslant \left(\frac{\omega(1)}{\tau(1)}\right)^{1/\rho},$$
$$\varlimsup_{n\to\infty}\frac{2E_n^{1/n}(f)}{|a_n|^{1/n}} \leqslant \left(\frac{\tau(1)}{\omega(1)}\right)^{1/\rho}.$$

Hence the result if $k = 1$. Similarly, we can prove it if $k \geqslant 2$.

Remark. We have from (27), for $k = 1$,

$$\limsup_{n\to\infty}\frac{2E_n^{1/n}(f)}{|a_n|^{1/n}} \leqslant \left(\frac{\tau(1)}{\omega(1)}\right)^{1/\rho}.$$

If f is such that $\tau(1) < 2\omega(1)$, then

$$\lim_{n\to\infty}(E_n(f)/|a_n|) = 0.$$

In other words, for functions f whose type is less than twice their lower type, $\lim_{n\to\infty}(E_n(f)/|a_n|) = 0$.

THEOREM 10. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function of logarithmic order ρ_l, with corresponding types τ_l, ω_l. Then, if $|a_n/a_{n+1}|$ and $E_n(f)/E_{n+1}(f)$ are nondecreasing for $n \geqslant n_0$, we have*

$$\left(\frac{\omega_l}{\tau_l}\right)^{1/(\rho_l-1)} \leqslant \varliminf_{n\to\infty}\frac{\log E_n(f)}{\log|a_n|} \leqslant 1 \leqslant \varlimsup_{n\to\infty}\frac{\log E_n(f)}{\log|a_n|} \leqslant \left(\frac{\tau_l}{\omega_l}\right)^{1/(\rho_l-1)}. \quad (30)$$

Proof. It is known, under our assumptions [3, Theorem 7] that

$$\lim_{n\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{(n/\rho_l)^{\rho_l}}{[(-\log E_n(f))/(\rho_l-1)]^{\rho_l-1}} = \begin{matrix}\tau_l\\ \omega_l\end{matrix}$$
$$= \lim_{n\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{(n/\rho_l)^{\rho_l}}{[(-\log|a_n|)/(\rho_l-1)]^{\rho_l-1}}. \quad (31)$$

With some manipulation of (31), one obtains the required result (30).

THEOREM 11. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *and* $g(z) = \sum_{k=0}^{\infty} b_k z^k$ *be two entire functions of the same positive order and of regular growth. There exists a sequence of positive integers* $\{n_p\}_1^{\infty}$, $n_p \to \infty$, *such that*

$$\lim_{p\to\infty} \frac{\log E_{n_p}(f)}{\log E_{n_p}(g)} = 1. \tag{32}$$

Proof. From [3, Theorems 1, 2a, 2b], [6, p. 44], and the existence of a sequence of positive integers $\{n_p\}$ such that $n_p \to \infty$ and

$$\lim_{p\to\infty} \frac{\log n_{p+1}}{\log n_p} = 1,$$

we infer

$$\lim_{p\to\infty} \frac{n_p \log n_p}{\log[1/E_{n_p}(f)]} = \rho = \lim_{p\to\infty} \frac{n_p \log n_p}{\log[1/E_{n_p}(g)]}.$$

Now we have the required result (32), as in the case of Theorem 6.

THEOREM 12. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z_k$ *and* $g(z) = \sum_{k=0}^{\infty} b_k z^k$ *be two entire functions of perfectly regular growth* (ρ, τ). *Then there exists a sequence of positive integers* $\{n_p\}_1^{\infty}$ $(n_p \to \infty)$ *such that*

$$\lim_{p\to\infty} \left(\frac{E_{n_p}(f)}{E_{n_p}(g)}\right)^{1/n_p} = 1. \tag{33}$$

Proof. This theorem follows, as Theorem 7, by using [3, Theorem 3]; hence we omit a detailed proof.

Example. Let $f(x) = e^{z\pi/4}$, $g(z) = \cos \pi z/4$. Then f and q are entire functions of perfectly regular growth $(1, \pi/4)$. It is known [2, p. 80] that

$$\frac{1}{2^n(n+1)!}(e^{-\pi/4}) \leqslant E_n(e^{z\pi/4}) \leqslant \frac{1}{2^n(n+1)!}(e^{\pi/4})$$

and

$$\frac{\pi^{2m+2}}{2^{6m+5}(2m+2)!} \leqslant E_{2m+1}\left(\cos\frac{\pi z}{4}\right) \leqslant \frac{\pi^{2m+2}}{2^{6m+5}(2m+2)!}.$$

From these inequalities one infers, taking $n_p \equiv 2p + 1$, that

$$\lim_{p\to\infty} \left(\frac{E_{n_p}(f)}{E_{n_p}(g)}\right)^{1/n_p} = 1.$$

THEOREM 13. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be two entire functions of index k, orders $\rho_f(k)$, $\rho_g(k)$ and lower orders $\lambda_f(k)$, $\lambda_g(k)$. If $E_n(f)/E_{n+1}(f)$ and $E_n(g)/E_{n+1}(g)$ are nondecreasing for $n \geqslant n_0$, then*

$$\frac{\lambda_f(k)}{\rho_g(k)} \leqslant \liminf_{n\to\infty} \frac{\log E_n(f)}{\log E_n(g)} \leqslant \frac{\lambda_f(k)}{\lambda_g(k)} \leqslant \limsup_{n\to\infty} \frac{\log E_n(f)}{\log E_n(g)} \leqslant \frac{\rho_f(k)}{\lambda_g(k)}. \tag{34}$$

Proof. We have from [3, Theorems 1, 2], under the additional assumptions that $E_n(f)/E_{n+1}(f)$ and $E_n(g)/E_{n+1}(g)$ are nondecreasing for $n \geqslant n_0$, that

$$\begin{matrix}\rho_f(k) \\ \lambda_g(k)\end{matrix} = \lim_{n\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{n l_k n}{\log[1/E_n(f)]}, \qquad \begin{matrix}\rho_g(k) \\ \lambda_g(k)\end{matrix} = \lim_{n\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{n l_k n}{\log[1/E_n(g)]}. \tag{35}$$

The required result follows from (35), as in the proof of Theorem 8.

EXAMPLE. Let $f(z) = e_k(z), g(z) = e_k(z^h)$; h is any positive integer (e.g., $e_2(z) = e^{e^z}$). Then

$$\lim_{n\to\infty} \left(\frac{\log E_n(f)}{\log E_n(g)}\right) = \frac{1}{h}.$$

THEOREM 14. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be two entire functions with index k, orders $\rho_f(k)$ and $\rho_g(k)$, and the associated numbers $\tau_f(k)$, $\omega_f(k)$ and $\tau_g(k)$, $\omega_g(k)$. If $E_n(f)/E_{n+1}(f)$ and $E_n(g)/E_{n+1}(g)$ are nondecreasing for $n \geqslant n_0$, then*

$$\begin{aligned}\frac{\omega_f(k)\,\rho_g(k)}{\tau_g(k)\,\rho_f(k)} &\leqslant \liminf_{n\to\infty} \frac{E_n(f)^{\rho_f(k)/n}}{E_n(g)^{\rho_g(k)/n}} \leqslant \frac{\omega_f(k)\,\rho_g(k)}{\omega_g(k)\,\rho_f(k)} \\ &\leqslant \limsup_{n\to\infty} \frac{\{E_n(f)\}^{\rho_f/n}}{\{E_n(g)\}^{\rho_g/n}} \leqslant \frac{\rho_f(k)\,\tau_g(k)}{\rho_f(k)\,\omega_g(k)}.\end{aligned} \tag{36}$$

Proof. By [3, Theorems 3, 4], we have, for $k = 1$:

$$\begin{aligned}\begin{matrix}\tau_f = \tau_f(k) \\ \omega_f = \omega_f(k)\end{matrix} &= \lim_{n\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{n}{\rho_f e} E_n(f)^{\rho_f/n}, \\ \begin{matrix}\tau_g \\ \omega_g\end{matrix} &= \lim_{n\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{n}{\rho_g e} E_n(g)^{\rho_g/n}.\end{aligned} \tag{37}$$

For $k \geqslant 2$,

$$\begin{aligned}\begin{matrix}\tau_f(k) \\ \omega_f(k)\end{matrix} &= \lim_{n\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} (l_{k-1} n)\, E_n(f)^{\rho_f(k)/n}, \\ \begin{matrix}\tau_g(k) \\ \omega_g(k)\end{matrix} &= \lim_{n\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} (l_{k-1} n)\, E_n(g)^{\rho_g(k)/n}.\end{aligned} \tag{38}$$

One derives the required result from (37) and (38).

Remark. Let $f(z) = e^z, g(z) = e^{2z}$. For these two functions, $\rho_f = 1$, $\rho_g = 1, \tau_f = \omega_f = 1, \tau_g = \omega_g = 2$. We have, from (36), for these functions,

$$\lim_{n\to\infty} \left(\frac{E_n^{1/n}(f)}{E_n^{1/n}(g)}\right) = \frac{1}{2}.$$

Hence

$$\lim_{n\to\infty} (E_n(f)/E_n(g)) = 0.$$

This result is sharper than the one obtained by Bernstein [2, Theorem 59].

THEOREM 15. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g(z) = \sum_{k=0}^{\infty} b_k z^k$ to two entire functions with index k and orders $\rho_f(k)$, $\lambda_f(k)$, $\rho_g(k)$, $\lambda_g(k)$. Assume that $E_n(f)/E_{n+1}(f)$ and $E_n(g)/E_{n+1}(g)$ are nondecreasing for $n \geqslant n_0$. If $\rho_f(k) = 0$ and $\lambda_l(f) > 1$, then*

$$\frac{\lambda_g(k)}{\rho_l(f) - 1} \leqslant \liminf_{n\to\infty} \frac{\log E_n(f)}{\log E_n(g)}$$

and (39)

$$\frac{\rho_g(k)}{\lambda_l(f) - 1} \leqslant \limsup_{n\to\infty} \frac{\log E_n(f)}{\log E_n(g)}.$$

If $\rho_g(k) = 0$ and $\lambda_f(k) > 0$, then

$$\lim_{n\to\infty} \frac{\log E_n(f)}{\log E_n(g)} \leqslant \min\left\{\frac{\lambda_l(g) - 1}{\lambda_f(k)}, \frac{\rho_l(g) - 1}{\rho_f(k)}\right\}$$

and

$$\limsup_{n\to\infty} \frac{\log E_n(f)}{\log E_n(g)} \leqslant \frac{\rho_l(g) - 1}{\lambda_f(k)}.$$

Proof. By [3, Theorems 1, 2, 5, 6]

$$\frac{\rho_f(k)}{\lambda_f(k)} = \lim_{n\to\infty} \frac{\sup}{\inf} \frac{n l_k n}{\log[1/E_n(f)]}, \qquad \frac{\rho_g(k)}{\lambda_g(k)} = \lim_{n\to\infty} \frac{\sup}{\inf} \frac{n l_k n}{\log[1/E_n(g)]}$$

$$\frac{\rho_l(f)}{\lambda_l(f)} = \lim_{n\to\infty} \frac{\sup}{\inf} \frac{\log n}{\log\{1/n \log(1/E_n(f))\}}, \tag{40}$$

$$\frac{\rho_l(g)}{\lambda_l(g)} = \lim_{n\to\infty} \frac{\sup}{\inf} \frac{\log n}{\log\{1/n \log(1/E_n(g))\}}.$$

(39) follows from (40) by some manipulations which we omit. The rest follows similarly.

THEOREM 16. *Let $f(x) = \sum_{k=0}^{\infty} a_k z^k$ (a_k real) be an entire function with index $k \geqslant 1$, order $\rho(k)$ and lower order $\lambda(k)$. If $E_n(f)/E_{n+1}(f)$ and $|a_n/a_{n+1}|$ are nondecreasing for $n \geqslant n_0$, then we have*

$$\frac{\lambda(k)}{\rho(k)} \leqslant \liminf_{n\to\infty} \frac{\log E_n^{1/n}(f)}{\log[E_{n+1}(f)/E_n(f)]} \leqslant 1 \leqslant \limsup_{n\to\infty} \frac{\log E_n^{1/n}(f)}{\log[E_{n+1}(f)/E_n(f)]} \leqslant \frac{\rho(k)}{\lambda(k)},$$

$$\frac{\lambda(k)}{\rho(k)} \leqslant \liminf_{n\to\infty} \frac{\log[E_{n+1}(f)/E_n(f)]}{\log |a_n|^{1/n}} \leqslant 1 \leqslant \limsup_{n\to\infty} \frac{n\log[E_{n+1}(f)/E_n(f)]}{\log |a_n|^{1/n}} \leqslant \frac{\rho(k)}{\lambda(k)}, \tag{41}$$

$$\frac{\lambda(k)}{\rho(k)} \leqslant \liminf_{n\to\infty} \frac{\log[E_{n+1}(f)/E_n(f)]}{\log |a_{n+1}/a_n|} \leqslant 1 \leqslant \limsup_{n\to\infty} \frac{\log[E_{n+1}(f)/E_n(f)]}{\log |a_{n+1}/a_n|} \leqslant \frac{\rho(k)}{\lambda(k)},$$

and

$$\frac{\lambda(k)}{\rho(k)} \leqslant \liminf_{n\to\infty} \frac{\log E_n^{1/n}(f)}{\log |a_{n+1}/a_n|} \leqslant 1 \leqslant \limsup_{n\to\infty} \frac{\log E_n^{1/n}(f)}{\log |a_{n+1}/a_n|} \leqslant \frac{\rho(k)}{\lambda(k)}.$$

Proof. By Lemmas 1, 2 and Theorems 3, 4,

$$\frac{\rho(k)}{\lambda(k)} = \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{nl_k n}{\log[1/E_n(f)]} = \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_k n}{\log[E_n(f)/E_{n+1}(f)]} \tag{42}$$

and

$$\frac{\rho(k)}{\lambda(k)} = \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{nl_k n}{\log |1/a_n|} = \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_k n}{\log |a_n/a_{n+1}|}.$$

(41) follows, using some manipulations, from (42).

The proof of the remaining assertions is similar and omitted.

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